Section 2.4

There are some functions whose limits do not exist yet still behave in predictable manners. We will see a way of describing such behaviors in this section.

Infinite Limits

The speed λ at which the edge of a a spinning Euler's Disk contacts the surface of the mirror is inversely proportional to the cube root of the remaining spinning time:

$$\lambda(t) = \frac{K'}{\sqrt[3]{t_f - t}}$$

where t_f is the time at which the disk comes to a complete stop.

What is the behavior of λ as t approaches t_f? $\lim_{t \to t_f^-} \lambda(t) =$

t	$\lambda(t)$
119	
119.9	
119.99	
119.999	

Definition 2.4.1 (Infinite Limits)**.** Suppose f is defined for all x near c with $x < c$. If $f(x)$ grows arbitrarily large for all $x < c$ sufficiently close (but not equal) to c, we write

$$\lim_{x \to c^-} f(x) = \infty.$$

If $f(x)$ is negative and grows arbitrarily large in magnitude for all $x < c$ sufficiently close (but not equal) to c, we write

$$\lim_{x \to c^-} f(x) = -\infty.$$

The right-sided limits are defined analogously.

If $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = \infty$, we write

$$\lim_{x \to c} f(x) = \infty.$$

If $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = -\infty$, we write

$$\lim_{x \to c} f(x) = -\infty.$$

In all cases, the limit does not exist.

Example 2.4.2. Let $f(x) = \dfrac{x+2}{(x^2-4)^2}$. Use the graph of f to find

1. $\lim\limits_{x \to 2^-} f(x)$

2. $\lim\limits_{x \to 2^+} f(x)$

3. $\lim\limits_{x \to 2} f(x)$

4. $\lim\limits_{x \to -2^-} f(x)$

5. $\lim\limits_{x \to -2^+} f(x)$

6. $\lim\limits_{x \to -2} f(x)$

Definition 2.4.3 (Vertical Asymptote)**.** For any infinite limit $\lim_{x\to c} f(x) = \pm\infty$, $\lim_{x\to c^+} f(x) = \pm\infty$, or $\lim_{x\to c^-} f(x) = \pm\infty$, the vertical line $x = c$ is a *vertical asymptote* of the graph of f.

Example 2.4.4. Consider $f(x) = \dfrac{x}{(x^2-1)^2}$. Use the graph of f to find

1. $\lim_{x\to 1^-} f(x)$

2. $\lim_{x\to 1^+} f(x)$

3. $\lim_{x\to -1} f(x)$

4. List all vertical asymptotes of f

Theorem 2.4.5 (Analytic Method)**.** Let $f(x) = \frac{h(x)}{g(x)}$ and suppose $\lim_{x\to c^-} h(x) = L \neq 0$ and $\lim_{x\to c^-} g(x) = 0$. If $f(x) > 0$ for all $x < c$ sufficiently close to c, then

$$\lim_{x\to c^-} f(x) = \infty.$$

If $f(x) < 0$ for all $x < c$ sufficiently close to c, then

$$\lim_{x\to c^-} f(x) = -\infty.$$

Right-sided limits can be analyzed analogously.

Example 2.4.6. Let $f(x) = \dfrac{x^3 + x^2 - 6x}{x^2 - x}$. Find

1. $\lim_{x \to 1^-} f(x)$

2. $\lim_{x \to 1^+} f(x)$

3. $\lim_{x \to 0} f(x)$

4. List all vertical asymptotes of f.

Example 2.4.7. Compute $\lim_{\theta \to 0} \cot \theta$

Example 2.4.8. Is it true if $f(x) = \frac{h(x)}{g(x)}$ and $\lim_{x \to c^+} h(x) = L \neq 0$ and $\lim_{x \to c^+} = 0$, then $\lim_{x \to c^+} f(x) = \pm\infty$?